JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 15, No. 5, September–October 1992

Approximate Pole Placement Approach

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A recently developed design method is applied to the benchmark problem. The design method may be viewed as a numerical loop-shaping method that also permits approximate assignment of the dominant poles of the closed-loop system. Solutions with different properties are obtained depending on the choice of closed-loop poles. Three different designs are given. The first is a a high-gain solution, the second gives a controller with relative degree 2, and the last one gives a controller with relative degree 1. The controllers satisfy the given design criteria. The chosen design method also makes it possible to tune a controller for structured parameter variations. A special controller, which admits a wide range of variations in the spring constant, is designed. It is also shown that the design method can easily be extended to deal with disturbances with known frequencies.

I. Introduction

MANY factors have to be considered in the design of a controller. Some key issues are command signal following, rejection of load disturbances, sensitivity to measurement noise, and model uncertainty.

For the example investigated in this paper, the emphasis is on rejection of load disturbances and parametric model uncertainty. Load disturbances enter as disturbance forces on the mechanical system; the uncertainty is variation of a spring constant. Sensor noise has been added to get a more realistic problem.

Control theory has attempted to formulate and solve idealized problems that focus on one or two of the issues just mentioned. This has led to design methods such as pole placement, linear quadratic Gaussian, predictive control, internal model control, quantitative control theory, H^∞, and parametric optimization. These methods are well described in textbooks (see Refs. 1–6). There is, however, no design methodology that covers all aspects, and control system design is, therefore, still a heuristic procedure. In this paper, we are using a new design method called approximative pole placement developed in Ref. 7. This method is inspired by the loop-shaping techniques due to Bode and Horowitz. The key idea is to replace the graphical constructs of Horowitz by an analytic method. This is done by formulating loop shaping as an optimization problem that can be solved analytically. An interesting side effect is that the method can also be interpreted as an approximate pole placement. A consequence of this is that the method makes sure that the dominating closed-loop poles have good values.

The paper is organized as follows. The design method is briefly described in Sec. II. In Sec. III, results of applying the method to the design problem are given. It is shown how solutions similar to previous designs are obtained by different choices of the closed-loop poles. It is also demonstrated that the previous designs can be improved. An interesting feature of the Horowitz design method is that it can explicitly cope with parametric uncertainty by shaping the ideal open-loop response. For gain variations, this is solved by Bode's ideal loop transfer function. In Sec. IV, it is shown that the problem of variations in the spring constant can be reduced to an equivalent problem with gain variations, and a solution is also provided. Section V considers the case with a disturbance of known frequency. It is shown that this case is easily dealt with. Conclusions are given in Sec. VI.

Received April 30, 1991; revision received Oct 30, 1991; accepted for publication Nov. 13, 1991.

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II. Approximate Pole Placement

The classical frequency domain design methods have many useful features. In these methods a plant model with uncertainty is given and specifications on the closed-loop system are stated in terms of a given closed-loop response. The method is a trial and error procedure. However, a significant drawback is that the methods rely on graphical methods for loop shaping. The idea of the approximate pole placement method is to develop a loop-shaping method that can be automated. This is obtained by formulating loop shaping as an optimization problem. There are many ways to do this. A detailed treatment is given in Lilja.[7] We will first present a method that is suitable for the design problem at hand. Consider a two-degree-of-freedom controller on the form

$$R(s)U = -S(s)Y + T(s)Y_{sp}$$

where y is the process output, u the control variable, and y_{sp} the setpoint. The polynomials R, S, and T are given by

$$R(s) = s^{nR} + r_1 s^{nR-1} + \cdots + r_{nR}$$
$$S(s) = s_0 s^{nS} + s_1 s^{nS-1} + \cdots + s_{nS}$$
$$T(s) = t_0 A_o(s)$$

The polynomial $A_o(s)$ is part of the specification that can be interpreted as an observer polynomial. Let the desired closed-loop transfer function be given by $G_m(s)$. The range of frequencies for which the closed-loop system should match G_m is also specified.

The closed-loop transfer function from setpoint y_{sp} to process output y is given by

$$G_{cl}(s) = \frac{G(s)T(s)}{R(s) + G(s)S(s)} \tag{1}$$

Introduce the relative closed-loop model error

$$E(s) = \frac{G_m(s) - G_{cl}(s)}{G_{cl}(s)}$$

A simple calculation gives

$$E(s) = -1 + F_R(s)\frac{R(s)}{t_0} + F_S(s)\frac{S(s)}{t_0} \tag{2}$$

where

$$F_R(s) = \frac{G_m(s)}{G(s)A_o(s)}$$

$$F_S(s) = \frac{G_m(s)}{A_o(s)}$$

It follows from Eq. (2) that the controller parametrization

$$\theta = \frac{1}{t_0}(1, r_1, \ldots, r_{nR}, s_0, \ldots, s_{nS})^T$$

gives linearity in $E(s)$. The closed-loop transfer function $G_{cl}(s)$ can be matched to the desired model $G_m(s)$ at some set of complex frequencies $Z = \{Z_i\}_{i=1}^N$ by minimizing the function

$$J = \sum_{i=1}^{N} |E(z_i)|^2$$

Since J is quadratic in θ, the minimization is carried out using the ordinary least-squares method. The set Z will be chosen as $\{-i\omega_1, i\omega_1, \ldots, -i\omega_M, i\omega_M\}$, which is also denoted as $\Omega = \{\omega_1, \ldots, \omega_M\}$.

The design method is straightforward. The user specifies the desired closed-loop transfer function G_m, the observer polynomial A_o, and a range of frequencies where a good fit is required. The controller is then obtained by a simple least-squares calculation. In the particular case when the desired closed-loop transfer function is a rational function, i.e.,

$$G_m(s) = \frac{B_m(s)}{A_m(s)}$$

the error E can be written as

$$E(s) = \frac{A(s)R(s) + B(s)S(s) - A_m(s)A_o(s)}{A_m(s)A_o(s)} \tag{3}$$

Notice that in an ordinary pole placement design the polynomials $R(s)$ and $S(s)$ are determined so that

$$A(s)R(s) + B(s)S(s) - A_o(s)A_m(s) = 0$$

(See Åström and Wittenmark.[1]) Equation (3) establishes the relation between the proposed design method and pole placement. If the loss function can be made equal to zero, the model is identical to pole placement. This will occur when $G_m(s)$ is rational and the controller has sufficiently high order.

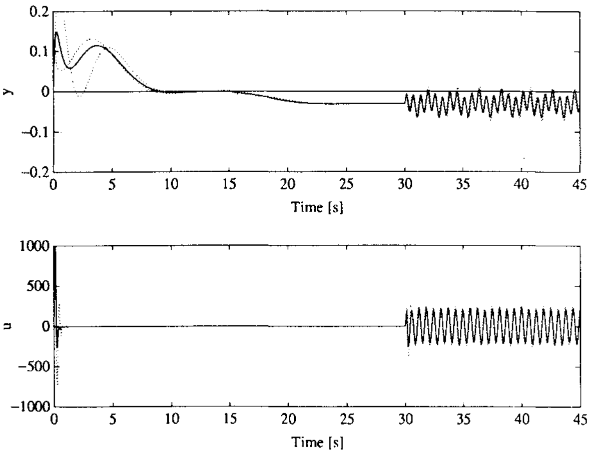

Fig. 1 Time responses to the disturbances of the closed-loop system obtained with design A1.

Notice that in this case it does not matter how the set Z is chosen, provided that it contains a sufficient number of points. The proposed method can, however, be used for low-order controllers and for nonrational reference models. In such a case, the set Z determines the frequency of the fit.

The fact that the error that is minimized in the design method can be written both as Eqs. (2) and (3) implies that the method attempts to obtain the desired closed-loop transfer function and also a desired closed-loop characteristic polynomial. This explains the name approximate pole placement.

III. Proposed Designs (Problem 1)

In this section, we will first show that the earlier designs can be obtained as approximate pole placement designs. The designs will first be carried out for a fixed value of the spring constant k. With the chosen ranges, it is straightforward to determine that the lowest value $k = 0.5$ is the worst case. It turns out that the designs are reasonably robust to variations in k. A method that specifically designs for robustness with respect to variations in k is given in Sec. IV.

Evaluation of the Designs

The behaviors of the closed-loop systems obtained with the different designs are illustrated in two ways. Time responses are generated from simulations performed as follows. Let t_m denote the simulation time. At time $t = 0$, an impulse is introduced in f_2. A step of height -0.05 appears in f_1 at $t = t_m/3$ and measurement noise $n(t) = 0.02(\sin 3t + \sin 10t)$ acts on the system from time $t = 2t_m/3$. The simulations are repeated for different values of the spring constant to illustrate the effects of changing this parameter. The solid curves will correspond to $k = 1$ and the dotted curves to $k = 0.5$ and 2. Transfer functions from disturbances to the output and the control signal will also be given.

Design A1

To obtain a design (A1) that is similar to the high bandwidth design in Ref. 8, the closed-loop model $G_m(s)$ is chosen as

$$G_m(s) = \frac{1}{s^4 + 3.346s^3 + 4.732s^2 + 3.346s + 1}$$

The pole excess and the order are the same as for the plant model. The poles are in a modified Butterworth pattern with radius 1 (which is of the same magnitude as the slowest poles in Fig. 7). The observer polynomial was chosen as

$$A_o(s) = s^3 + 60s^2 + 1800s + 2.7 \cdot 10^4$$

with zeros in an ordinary Butterworth pattern with radius 30. The controller can thus be interpreted as a fast observer combined with a state feedback that gives damped closed-loop poles in the neighborhood of the open-loop poles. Computing controllers of different orders that minimize the loss function showed that a controller of third order is required. The loss function will then be zero and the method is equivalent to ordinary pole placement. The controller obtained is

$$R(s) = (s + 33.13)(s^2 + 30.21s + 1003)$$

$$S(s) = 194117(s + 0.4005)(s^2 + 0.636s + 0.6947)$$

$$T(s) = 2(s + 30)(s^2 + 30s + 900)$$

This gives a high-gain controller whose properties are similar to design A. Notice that the low-frequency gain of the controller is 1.7, but that the high-frequency gain is extremely high: $1.5 \cdot 10^5$. The reason for this large gain is due to the fast observer. The high-frequency gain decreases significantly with a slower observer.

The time response of the closed-loop system is shown in Fig. 1. The large control signals due to the impulse disturbance and the noise are what have to be paid to achieve the insensitivity to changes in the spring constant k. The closed-loop system is stable for spring constants in the range $0.16 \leq k \leq 3.8$.

Design B1

Design A1 has an unrealistically high gain. To obtain a design with more moderate gains, we will now consider a design that is similar to the one proposed in Ref. 9. The design model $G_m(s)$ is then chosen as

$$G_m(s) = \frac{0.2401}{s^4 + 1.829s^3 + 1.673s^2 + 0.8963s + 0.2401}$$

This system has poles in a standard Butterworth pattern with radius 0.7. The observer polynomial is chosen as

$$A_o(s) = s^3 + 1.4s^2 + 0.98s + 0.343$$

A third-order controller with relative degree 2

$$R(s) = (s + 1.082)(s^2 + 1.221s + 1.035)$$
$$S(s) = 0.677903(s + 0.1517)$$
$$T(s) = 0.299791(s + 0.7)(s^2 + 0.7s + 0.49)$$

was computed using approximate pole placement with approximation frequencies $\Omega = \{0.01, 0.1, 0.2, 0.3, 0.4\}$. A simulation of the closed-loop system for the nominal value of the spring constant and for the extreme values are shown in Fig. 2.

Further experimentation revealed that it is difficult to damp the oscillatory poles with a controller of relative degree 2. The closed-loop system is stable for spring constants in the range $0.34 \leq k \leq \infty$.

Design C1

Since it was difficult to damp oscillatory poles with a controller having relative degree 2, a design that gave a controller with relative degree 1 was chosen. The system obtained is similar to that in Ref. 10. Some experimentation showed that the robustness to variations in the spring constant could be improved by designing for a nominal case with $k = 0.7$. The closed-loop model G_m and the observer polynomial A_o were chosen as

$$G_m(s) = \frac{1}{(s^2 + 0.7s + 0.25)(s^2 + 2.8s + 4)}$$

$$A_o(s) = (s + 0.8)(s^2 + 1.131s + 0.64)$$

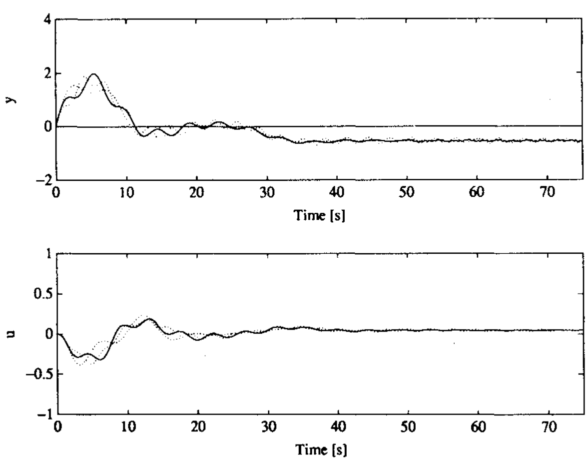

Fig. 2 Time responses to the disturbances of the closed-loop system obtained with design B1.

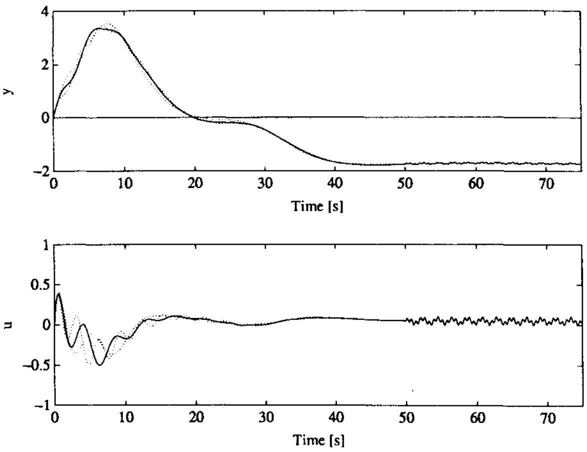

Fig. 3 Time responses to the disturbances of the closed-loop system obtained with design C1.

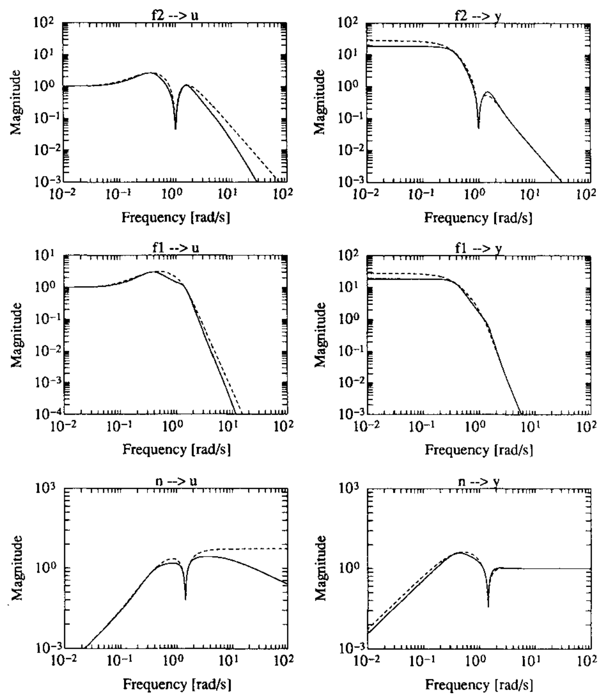

Fig. 4 Transfer functions characterizing the transmission from disturbance to outputs and control signals for design C1 (solid) and design C (dashed).

Notice the comparatively slow poles in the closed-loop model that are introduced to increase the robustness. The approximate pole placement method with $\Omega = \{0.01, 0.1, 0.2, 0.3, 0.4\}$ was utilized to compute the controller

$$R(s) = (s + 7.576)(s^2 + 3.063s + 3.571)$$
$$S(s) = -24.6668(s - 0.4957)(s + 0.1172)$$
$$T(s) = 2.79831(s + 0.8)(s^2 + 1.131s + 0.64)$$

The controller has a zero in the right half-plane. The reason for this is that the controller has to decrease the response speed of the system in order to achieve the specified closed-loop model. The simulation in Fig. 3 shows that the closed-loop system satisfies the settling time requirements for all values of the spring constant. The closed-loop system obtained was then stable for $0.38 \leq k \leq 14$. The properties of the closed-loop system are further illustrated in Fig. 4, which shows the transfer functions from f_1, f_2, and n to u and y for design C1. For comparison, the curves obtained with Ly's[10] controller (design C) are also shown in the figure (dashed curves).

IV. Gain Equivalence

The problem of designing a feedback system that is invariant to variations in the gain of a plant was formulated and solved by Bode.[11] His solution can be expressed very simply in terms of loop shaping. The problem is to find a compensator so that the Nyquist curve of the loop transfer function is approximately a straight half-line with argument $\pi + \phi_m$, where ϕ_m is the phase margin. Such a transfer function is called the ideal cutoff characteristics by Bode. The design gives a closed-loop system whose bandwidth changes with process gain, but the phase margin remains invariant. An invariant response is obtained simply by cascading with a sufficiently slow filter. This particular version of loop shaping can be obtained by approximate pole placement as is shown in Lilja.[7] It should be noticed that the Horowitz Quantitative Feedback Theory (QFT) method is a generalization of this idea.

In this particular example, there is one parameter that represents plant uncertainty. Bode's design method can be applied if the model can be rewritten so that the parameter appears as a gain variation. To do this we proceed as follows. For a given feedback $S(s)/R(s)$, the closed-loop poles are the roots of the polynomial P given by

$$P(s) = A(s)R(s) + B(s)S(s)$$

With a proper controller, the polynomial P can be written as

$$P(s) = s^4R(s) + k[2s^2R(s) + S(s)]$$

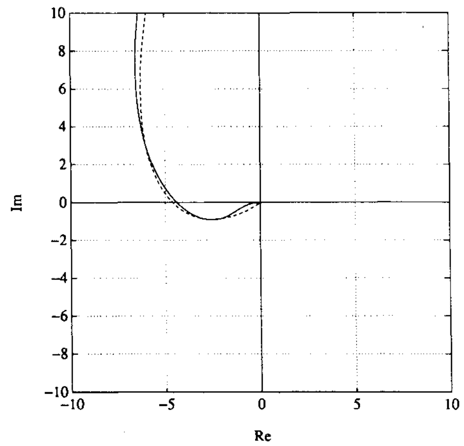

Fig. 5 Actual loop gain (solid) and ideal loop gain (dashed).

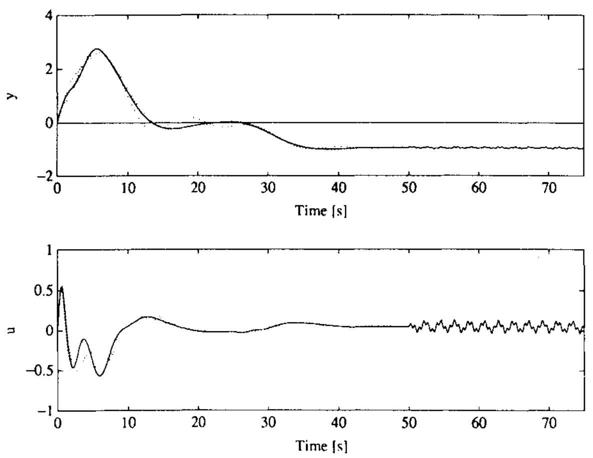

Fig. 6 Time response for closed-loop system.

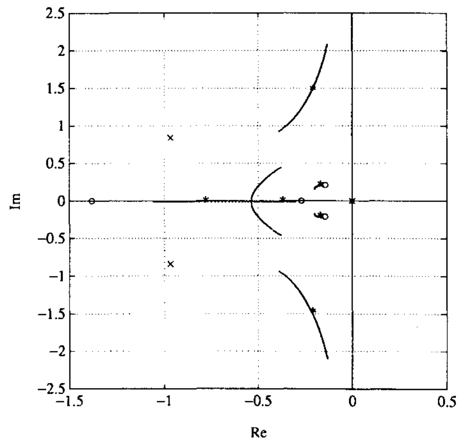

Fig. 7 Root locus with respect to k in the GE design where $0.5 \le k \le 2$. The closed-loop poles are also plotted for $k = 0(\times)$, $k = 1(*)$, and $k = \infty(\circ)$.

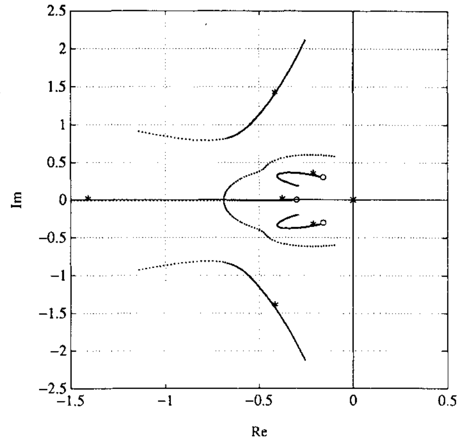

Fig. 8 Root locus with respect to k in design C1 where $0.5 \le k \le 2$. The closed-loop poles are also plotted for $k = 0(\times)$, $k = 1(*)$, and $k = \infty(\circ)$.

The spring constant k can thus be interpreted as a gain in front of a virtual loop transfer function L given by

$$L(s) = \frac{2s^2R(s) + S(s)}{s^4R(s)} = \frac{2s^2 + [S(s)/R(s)]}{s^4} \tag{4}$$

A loop-shaping procedure can now be carried out where the goal is to achieve insensitivity to gain variations. One example of loop transfer function of Bode's class is

$$\hat{L}(s) = \frac{k}{s^\alpha}$$

where $1 < \alpha < 2$. The phase margin of such a transfer function is given by

$$\phi_m = \frac{(2 - \alpha)\pi}{2}$$

independently of the value of k. This loop transfer function can only be approximated in a small frequency interval. If deg R = deg S in Eq. (4), it is seen that $L(s) \sim s^{-4}$ as $s \to 0$. To get a more realistic loop transfer function, $\hat{L}(s)$ is modified to

$$\hat{L}(s) = \frac{k_1}{s^{\alpha_1}} + \frac{k_2}{s^{\alpha_2}} + \frac{k_3}{s^{\alpha_3}} \tag{5}$$

where $1 < \alpha_1 < 2$, $\alpha_2 = 3$, and $\alpha_3 = 4$. The coefficients in the polynomials R and S were computed by the least-squares

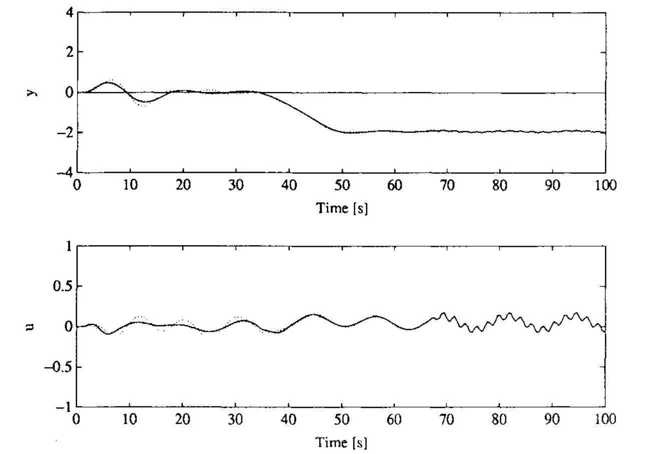

Fig. 9 Time responses to the disturbances of the closed-loop system obtained with design C3.

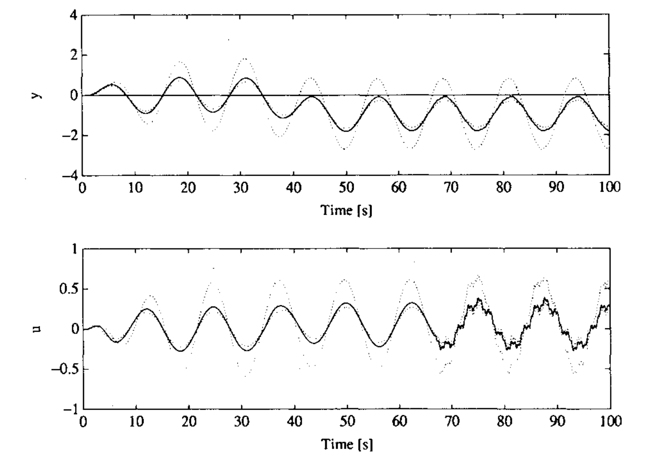

Fig. 10 Time responses to the disturbances of the closed-loop system obtained with design C1.

method to fit L in Eq. (4) to $\hat{L}$ in Eq. (5) at certain sets of frequencies Ω. The following parameter values were obtained for a second-order controller:

$$R(s) = (s^2 + 1.936s + 1.644)$$

$$S(s) = -1.47365(s - 0.3753)(s + 0.08676)$$

$$T(s) = S(s)$$

Notice that the controller has a zero in the right half-plane. Figure 5 shows the Nyquist curves for L and $\hat{L}$ for $k_1 = 1.67$, $k_2 = 0.292$, and $k_3 = 0.0252$ for $\hat{L}$ and $\Omega = \{0.1, 0.2, 0.3, 0.4, 0.6\}$.

The time response for the closed-loop system is shown in Fig. 6. The response is slower than for design C. This is the penalty for obtaining robustness. The root locus with respect to the spring constant k is shown in Fig. 7. Note that the pole at 0 (for $k = 0$) has multiplicity 4. Figure 7 shows that the closed-loop system is stable for the requested range of the spring constant $0.5 \leq k \leq 2$. An investigation shows that the closed-loop system is actually stable for $k \geq 0.23$. The root locus in Fig. 7 should be compared to the corresponding root locus for design C1 (Fig. 8).

V. Proposed Design (Problem 3)

In problem 3, a sinusoidal disturbance $d(t) = \sin 0.5t$ is introduced in the output signal. This puts the requirement that $\pm 0.5i$ must be included in the poles of the feedback compensator. A new design (C3) is done by modifying design C1. The controller for design C3 is chosen to be of sixth order. The nominal value of the spring constant is chosen to be $k = 0.7$, as in design C1. The closed-loop model G_m and the observer polynomial A_o are modified to

$$G_m(s) = \frac{0.9375}{(s + 2.5)(s + 1.5)(s^2 + 0.9s + 0.25)}$$

$$A_o(s) = (s^2 + 0.5s + 0.25)(s^2 + 1.12s + 0.64) \times (s^2 + 2.16s + 1.44)$$

Using the approximate pole placement method with $\Omega = \{0.01, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6\}$ gives the controller

$$R(s) = (s^2 + 0.01s + 0.25)(s^2 + 5.902s + 9.369) \times (s^2 + 2.768s + 5.068)$$

$$S(s) = 0.107675(s - 321.8)(s - 0.4402)(s + 0.07936) \times (s^2 - 0.1043s + 0.2549)$$

$$T(s) = 1.33929(s^2 + 0.5s + 0.25)(s^2 + 2.16s + 1.44) \times (s^2 + 1.12s + 0.64)$$

The time responses are simulated with the same input signals as before, with one exception: The impulse in f_2 at $t = 0$ is replaced by the signal $f_2(t) = \cos 0.5t$. The result for design C3 is shown in Fig. 9. This should be compared with the corresponding simulation for design C1 (Fig. 10).

VI. Conclusions

It has been demonstrated that solutions to the problem can be generated by approximate pole placement. This design procedure can be viewed as a numerical loop-shaping procedure where the specifications are given in terms of the desired loop transfer function, an observer polynomial, and a set of frequencies. It has been demonstrated that improved designs can be obtained using the procedure.

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